The Remainder and Factor Theorems

Exercises

1. a. Without dividing, determine the remainder when $x^3 - 4x^2 + 7x - 6$ is divided by
   i. $x - 1$
   ii. $x + 3$
   iii. $x - 2$
   iv. $2x - 1$

   b. Are any of the linear divisors in part a) factors of $x^3 - 4x^2 + 7x - 6$?

2. a. When $x^3 + kx^2 - 4x + 2, k \in \mathbb{R}$, is divided by $x + 2$, the remainder is 26. Find k.
   b. The remainder is 4 when $2x^2 - 3x + 5$ is divided by $x + k, k \in \mathbb{R}$. Find k.

3. a. Without dividing, determine which of the following are factors of $2x^4 - 15x^3 + 11x^2 + 78x - 40$?
   i. $x + 1$
   ii. $x + 2$
   iii. $x - 5$
   iv. $x + 4$

   b. Find the remaining factors of the polynomial $2x^4 - 15x^3 + 11x^2 + 78x - 40$.

4. Determine the value(s) of $k, k \in \mathbb{R}$:
   a. if $x - 5$ is a factor of $x^3 + 2x^2 + kx + 30$
   b. if $2x + 3$ is a factor of $2x^3 + kx^2 - 2x + 15$
   c. if $x + k$ is a factor of $2x^2 + kx - 9$
   d. if $2x + 1$ and $x - k$ are factors of $nx^2 - 9x - n, n \in \mathbb{R}$
5. State the equation of any cubic polynomial that has a remainder of -6 when divided by $x + 3$.
6. The polynomial $2x^3 + px^2 + qx + 12, p, q \in \mathbb{R}$, has a factor of $x + 3$ and a remainder of -10 when divided by $x - 2$. Find p and q.
7. For what value(s) of $k, k \in \mathbb{R}$, does the function $f(x) = x^3 + 6x^2 + kx - 4$ have the same remainder when divided by either $x - 1$ or $x + 2$?
8. Find the value of a and b if $x^2 - 5x + 4$ is a factor of the polynomial $2x^3 + ax^2 + bx - 4$. Express the polynomial in factored form.
9. Given that a and b are non-zero integers and n is a positive integer:
   a. for what values of n is $(a - b)$ a factor of $a^n - b^n$
   b. for what values of n is $(a + b)$ a factor of $a^n + b^n$
10. A polynomial $P(x)$ has a remainder of 3 when divided by $x - 2$ and a remainder of -5 when it is divided by $x + 2$. Determine the remainder when the polynomial is divided by $x^2 - 4$.

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Partial Solutions

1. There is no solution provided for this question.

2. a. Let $P(x) = x^3 + kx^2 - 4x + 2, k \in \mathbb{R}$. By the remainder theorem,
$$\begin{aligned} P(-2) &= (-2)^3 + k(-2)^2 - 4(-2) + 2 \\ 26 &= 2 + 4k \end{aligned}$$
Therefore, $k = 6$.

   b. Let $P(x) = 2x^2 - 3x + 5$. By the remainder theorem,
$$\begin{aligned} P(-k) &= 2k^2 + 3k + 5 \\ 4 &= 2k^2 + 3k + 5 \\ 0 &= 2k^2 + 3k + 1 \\ 0 &= (2k + 1)(k + 1) \end{aligned}$$
Therefore, $k = -\frac{1}{2}, -1$

3. There is no solution provided for this question.

4. By the factor theorem, $(ax + b)$ is a factor of a polynomial $P(x)$ if $P\left(-\frac{b}{a}\right) = 0$.

   a.
$$\begin{aligned} 0 &= P(5) \\ 0 &= (5)^3 + 2(5)^2 + k(5) + 30 \\ 0 &= 205 + 5k \\ \therefore k &= -41 \end{aligned}$$

   b.
$$\begin{aligned} 0 &= P\left(-\frac{3}{2}\right) \\ 0 &= 2\left(-\frac{3}{2}\right)^3 + k\left(-\frac{3}{2}\right)^2 - 2\left(-\frac{3}{2}\right) + 15 \\ 0 &= \frac{45}{4} + \frac{9k}{4} \\ \therefore k &= -5 \end{aligned}$$

   c.
$$\begin{aligned} 0 &= P(-k) \\ 0 &= 2(-k)^2 + k(-k) - 9 \\ 0 &= k^2 - 9 \\ 0 &= (k + 3)(k - 3) \\ \therefore k &= \pm 3 \end{aligned}$$

   d.
$$\begin{aligned} 0 &= P\left(-\frac{1}{2}\right) \\ 0 &= \frac{n}{4} + \frac{9}{2} - n \\ -\frac{9}{2} &= \frac{-3n}{4} \\ 36 &= 6n \\ \therefore n &= 6 \\ P(x) &= 6x^2 - 9x - 6 \\ &= (6x + 3)(x - 2) \\ &= 3(2x + 1)(x - 2) \end{aligned}$$
Therefore, $k = 2$.
Note: Using the factor theorem to determine k, $P(k) = 0$ since $(x - k)$ is a factor. So
$$\begin{aligned} P(k) &= 0 \\ 6k^2 - 9k - 6 &= 0 \\ (6k + 3)(k - 2) &= 0 \\ \therefore k &= -\frac{1}{2}, 2 \end{aligned}$$
However, when $k = -\frac{1}{2}$, the quadratic is of the form $a(2x + 1)\left(x + \frac{1}{2}\right) = a\left(2x^2 + 2x + \frac{1}{2}\right)$, which is not of the form $nx^2 + 9x - n$ for any $n \in \mathbb{R}$. Therefore, $k \neq -\frac{1}{2}$. $k = 2$ is the only solution.

5. There is no solution provided for this question.

6. Let $P(x) = 2x^3 + px^2 + qx + 12, p, q \in \mathbb{R}$. Since $x + 3$ is a factor,
$$\begin{aligned} P(-3) &= 0 \\ 2(-3)^3 + p(-3)^2 + q(-3) + 12 &= 0 \\ -42 + 9p - 3q &= 0 \\ 3p - q &= 14 \end{aligned} \tag{1}$$

Since $P(x)$ has a remainder of -10 when divided by $x-2$,

$$\begin{aligned} P(2) &= -10 \\ 2(2)^3 + 4p + 2q + 12 &= -10 \\ 28 + 4p + 2q &= -10 \\ 4p + 2q &= -38 \\ 2p + q &= -19 \qquad (2) \end{aligned}$$

Adding (2) to (1) gives $5p = -5, \therefore p = -1$. Substituting into (2), $2(-1) + q = -19, \therefore q = -17$.

7. There is no solution provided for this question.

8. Observe that $d(x) = x^2 - 5x + 4$ factors into $d(x) = (x-4)(x-1)$. Thus if $d(x)$ is a factor of $P(x) = 2x^3 + ax^2 + bx - 4$, then $(x-4), (x-1)$ are factors of $P(x)$. In particular, we have $P(4) = 0$ and $P(1) = 0$.

$$\begin{aligned} P(4) &= 0 \\ 2(4)^3 + a(4)^2 + b(4) - 4 &= 0 \\ 124 + 16a + 4b &= 0 \\ 4a + b &= -31 \qquad (1) \end{aligned}$$

and

$$\begin{aligned} P(1) &= 0 \\ 2(1)^3 + a(1)^2 + b(1) - 4 &= 0 \\ a + b &= 2 \qquad (2) \end{aligned}$$

Subtracting (2) from (1) gives $3a = -33$ so $a = -11$.

Substituting $a = -11$ into (2), $-11 + b = 2, \therefore b = 13$. Thus $P(x) = 2x^3 - 11x^2 + 13x - 4$.

Since $(x-4), (x-1)$ are factors of $P(x)$, and $P(x)$ is degree 3, we know $2x^3 - 11x^2 + 13x - 4 = (x-4)(x-1)(cx+d)$.

Considering the constant term, we see $(-4)(-1)d = -4, \therefore d = -1$; considering the leading coefficient, we see $c = 2$.

Therefore $P(x) = (x-1)(x-4)(2x-1)$.

9. a. For all $n,\ n \in \mathbb{Z},\ n > 0$. Let $P(a) = a^n - b^n$. $P(b) = b^n - b^n = 0$ for all $n \in \mathbb{Z},\ n > 0$. Therefore, by the factor theorem, $(a-b)$ is a factor of $a^n - b^n$ for all $n \in \mathbb{Z},\ n > 0$.

   b. For $n = 2k - 1$, where $k \in \mathbb{Z},\ k > 0$. Let $P(a) = a^n + b^n$. Now $P(-b) = (-b)^n + b^n$ will equal zero for all odd integer values of n. Therefore, by the factor theorem, $(a+b)$ is a factor of $a^n + b^n$ when $n = 2k-1,\ k \in \mathbb{Z},\ k > 0$.

10. When $P(x)$ is divided by $x-2$ the remainder is 3; therefore $P(2) = 3$.

When $P(x)$ is divided by $x+2$ the remainder is -5; therefore $P(-2) = -5$.

When dividing by $x^2 - 4$ the remainder is possibly linear or constant, since the remainder must have degree less than the degree of the divisor. Let the remainder be $ax + b$. Then

$$\begin{aligned} P(x) &= (x^2 - 4)q(x) + (ax + b) \\ &= (x-2)(x+2)q(x) + (ax+b) \end{aligned}$$

Since $P(2) = 3$,

$$\begin{aligned} P(2) &= 3 \\ (2-2)(2+2)q(x) + (2a+b) &= 3 \\ 2a + b &= 3 \qquad (1) \end{aligned}$$

Since $P(-2) = -5$,

$$\begin{aligned} P(-2) &= -5 \\ (-2-2)(-2+2)q(x) + (-2a+b) &= -5 \\ -2a + b &= -5 \qquad (2) \end{aligned}$$

Adding (1) to (2) gives $2b = -2$ so $b = -1$. Substituting $b = -1$ into (2) gives $-2a - 1 = -5$ so $a = 2$.

Therefore the remainder is $2x - 1$ when the polynomial is divided by $x^2 - 4$.